Limits: A Numerical and Graphical Approach

OBJECTIVE

• Find limits of functions, if they exist, using numerical or graphical methods.
Question: As the input $x$ gets “closer” to 3, what happens to the value of $f(x)$?

- Analyze it graphically and numerically
**Answer:** As $x$ gets “closer” to 3, $f(x)$ gets closer to 6.

We write

$$\lim_{x \to 3} f(x) = 6$$
DEFINITION (informal):

The expression

$$\lim_{x \to a} f(x) = L$$

means “as $x$ gets ‘closer’ to the number $a$, $f(x)$ gets closer to the number $L$”
Comments

1. $x$ never reaches $a$
2. The value of $f(a)$ does not matter ($f(a)$ need not even be defined)
3. $x$ can approach from either direction, or both directions
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Example: Consider the function $H(x)$ graphed below.

Question: If $x$ is less than 1, but gets closer to 1, what happens to the value of $H(x)$?

Answer: $H(x)$ gets closer to 4

We write:

$$\lim_{x \to 1^-} H(x) = 4$$

Called a limit from the left.
Example: Consider the function $H(x)$ graphed below

Question: If $x$ is greater than 1, but gets closer to 1, what happens to the value of $H(x)$?

Answer: $H(x)$ gets closer to $-2$

We write:

$$\lim_{{x \to 1^+}} H(x) = -1$$

Called a limit from the right
THEOREMS

1. \( \lim_{{x \to a}} f(x) = L \) if and only if
   \[
   \lim_{{x \to a^-}} f(x) = L \quad \text{and} \quad \lim_{{x \to a^+}} f(x) = L
   \]

2. If \( \lim_{{x \to a^-}} f(x) \neq \lim_{{x \to a^+}} f(x) \), then \( \lim_{{x \to a}} f(x) = L \) does not exist.
Example: Consider the function $H(x)$ graphed below

$$\lim_{x \to 1^-} H(x) = 4, \quad \lim_{x \to 1^+} H(x) = -1$$

Question: Does $\lim_{x \to 1} f(x)$ exist?

Answer: NO, since the limit from the left does not equal the limit from the right
Example: Consider the function $H(x)$ graphed below.

Question: Does $\lim_{x \to -3} f(x)$ exist?

Answer: Yes, $\lim_{x \to -3} f(x) = -4$
Example: Calculate the following limits based on the graph of $f$

\[ \lim_{x \to 2^-} f(x) = 3 \]
\[ \lim_{x \to 2^+} f(x) = 3 \]
\[ \lim_{x \to 2} f(x) = 3 \]
For Exercises 43–52, use the following graph of $H$ to find each limit. When necessary, state that the limit does not exist.

43. $\lim_{x \to -3} H(x) = 0$
44. $\lim_{x \to -2^-} H(x) = 1$
45. $\lim_{x \to -2^+} H(x) = 1$
46. $\lim_{x \to -2} H(x) = 1$
47. $\lim_{x \to 1^-} H(x) = 4$
48. $\lim_{x \to 1^+} H(x) = 2$
49. $\lim_{x \to 1} H(x)$ Does not exist
50. $\lim_{x \to 3^-} H(x) = 1$
51. $\lim_{x \to 3^+} H(x) = 1$
52. $\lim_{x \to 3} H(x) = 1$
Example: Consider the function \( f(x) = \frac{1}{x} \)

Find the following limits:

a) \( \lim_{x \to 0^-} f(x) \)

b) \( \lim_{x \to 0^+} f(x) \)

c) \( \lim_{x \to 0} f(x) \)

graphically and numerically
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Solutions:

a) \( \lim_{x \to 0^-} f(x) = -\infty \)

**Means:** As \( x \) gets closer to 0 from the left, \( f(x) \) gets more negative

b) \( \lim_{x \to 0^+} f(x) = \infty \)

**Means:** As \( x \) gets closer to 0 from the right, \( f(x) \) gets more positive

c) \( \lim_{x \to 0} f(x) \) Does not exist
Important Point: $\infty$ is not a number!

- A number is a destination on the number line
- $\infty$ is not a destination, it is a journey
Example: Consider the function \( f(x) = \frac{1}{x} \)

Find the following limits:

a) \( \lim_{{x \to -\infty}} f(x) \)

b) \( \lim_{{x \to \infty}} f(x) \)

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Solutions:

a) \( \lim_{x \to -\infty} f(x) = 0 \)

\[ \begin{align*}
\text{Means: } & \text{As } x \text{ gets more negative,} \\
& f(x) \text{ gets closer to 0}
\end{align*} \]

b) \( \lim_{x \to \infty} f(x) = 0 \)

\[ \begin{align*}
\text{Means: } & \text{As } x \text{ gets more positive,} \\
& f(x) \text{ gets closer to 0}
\end{align*} \]
For Exercises 53–62, use the following graph of \( f \) to find each limit. When necessary, state that the limit does not exist.

53. \( \lim_{x \to -1} f(x) = 1 \)
54. \( \lim_{x \to 2} f(x) = -1 \)
55. \( \lim_{x \to -3} f(x) \) Does not exist
56. \( \lim_{x \to 0} f(x) = 2 \)
57. \( \lim_{x \to 3} f(x) = 0 \)
58. \( \lim_{x \to 1} f(x) \) Does not exist
59. \( \lim_{x \to -4} f(x) = 3 \)
60. \( \lim_{x \to -2} f(x) = 0 \)
61. \( \lim_{x \to \infty} f(x) = 1 \)
62. \( \lim_{x \to -\infty} f(x) = 2 \)
**Taxicab fares.** In New York City, taxicabs charge passengers $2.50 for entering a cab and then $0.40 for each one-fifth of a mile (or fraction thereof) traveled. (There are additional charges for slow traffic and idle times, but these are not considered in this problem.) If $x$ represents the distance traveled in miles, then $C(x)$ is the cost of the taxi fare, where

\[
C(x) = \begin{cases} 
2.50, & \text{if } x = 0, \\
2.90, & \text{if } 0 < x \leq 0.2, \\
3.30, & \text{if } 0.2 < x \leq 0.4, \\
3.70, & \text{if } 0.4 < x \leq 0.6, \\
\end{cases}
\]

and so on. The graph of $C$ is shown below. (Source: New York City Taxi and Limousine Commission.)
Using the graph of the taxicab fare function, find each of the following limits, if it exists.

81. \( \lim_{x \to 0.25^-} C(x), \lim_{x \to 0.25^+} C(x), \lim_{x \to 0.25} C(x) = 3.30, 3.30, 3.30 \)

82. \( \lim_{x \to 0.2^-} C(x), \lim_{x \to 0.2^+} C(x), \lim_{x \to 0.2} C(x) = 2.90, 3.30, \text{Does not exist} \)
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Homework

• 1.1 HW